L_2 -BOUNDEDNESS OF THE GAMMA-PRODUCT TRANSFORM

S.B. Yakubovich

Department of Pure Mathematics, Faculty of Science, University of Porto
 Campo Alegre str. 687, 4169-007 Porto, Portugal
 (e-mail: syakubov@fc.up.pt)

Abstract. We prove the Plancherel theorem and establish related L_2 -properties for the following singular integral transformation associated with the product of Euler's Gamma-functions:

$$[\mathcal{G}f](x) = \text{P.V.} \int_0^\infty \Gamma(i(x+\tau))\Gamma(i(x-\tau))f(\tau) d\tau, \quad x \in \mathbb{R}.$$

The results are based on the corresponding properties for the Kontorovich – Lebedev and Mellin transformations. Necessary and sufficient conditions of the analytic continuability of the Gamma-product transform to the right half-plane are derived.

Keywords: Euler Gamma-function, Bessel functions, Mellin–Barnes integrals, Kontorovich–Lebedev transform, singular integral.

Received 15 01 2006

1. INTRODUCTION AND PRELIMINARY RESULTS

Let $f: \mathbb{R}_+ \rightarrow \mathbb{C}$ be a measurable function. We deal with the integral transform

$$[\mathcal{G}f](x) = \text{P.V.} \int_0^\infty \Gamma(i(x+\tau))\Gamma(i(x-\tau))f(\tau) d\tau, \quad x \in \mathbb{R}, \quad (1.1)$$

which we will call the Gamma-product transformation. Here $\Gamma(z)$ is Euler's Gamma-function [1], and i is the imaginary unit. A complex extension of the Gamma-product transformation (1.1) has been introduced recently by the author [7], and analytic properties in the Hardy spaces were investigated. Since $\Gamma(z) \sim \frac{1}{z}$, $z \rightarrow 0$ [1], we have that, at the point $\tau = |x|$, integral (1.1) is understood in the Cauchy principal-value sense. For instance, similarly to the case of the Hilbert transform (cf. [5]), one can show that the singular integral in (1.1) is well-defined for bounded functions on $\mathbb{R}_+$ satisfying Hölder's condition with exponent $\lambda \in]0, 1]$ at neighborhoods of points $\tau = \pm x$. We note that, in the sequel, we use the weighted Lebesgue spaces $L_p(\Omega; \omega(x) dx)$ with respect to the measure $\omega(x) dx$ equipped with the norm

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \omega(x) dx \right)^{1/p}, \quad 1 \leq p < \infty, \quad (1.2)$$

$$\|f\|_\infty = \operatorname{ess\,sup} |f(x)|. \quad (1.3)$$

Our goal is to consider the L_2 -boundedness of the Gamma-product transform and to prove an analogue of the Plancherel theorem for this case. Our results are based on the L_2 -theory for the Kontorovich–Lebedev transform (see [4], [6])

$$[KLf](x) = \operatorname{l.i.m.}_{N \rightarrow \infty} \int_0^N K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau, \quad x > 0, \quad (1.4)$$

where $K_{2i\tau}(2\sqrt{x})$ is the modified Bessel function [2], and the limit in (1.4) is taken with respect to the norm in the space $L_2(\mathbb{R}_+; x^{-1} dx)$. This function is real-valued and can be given by the Mellin–Barnes integral (see [6, relation (1.113)])

$$K_{2i\tau}(2\sqrt{x}) = \frac{1}{4\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau) \Gamma(s-i\tau) x^{-s} ds, \quad (1.5)$$

where $x > 0$, $\gamma > 0$, $\tau \in \mathbb{R}$. As is known [3], [5], the theory of the Mellin–Barnes integrals is based on the Mellin direct and inverse transforms defined by the formulas

$$f^{\mathcal{M}}(s) = \int_0^\infty f(x) x^{s-1} dx, \quad (1.6)$$

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^{\mathcal{M}}(s) x^{-s} ds, \quad s = \gamma + it, \quad x > 0, \quad (1.7)$$

where integrals (1.6)–(1.7) exist as Lebesgue integrals or, in particular, in mean with respect to the norm of the spaces $L_2(\gamma - i\infty, \gamma + i\infty)$ and $L_2(\mathbb{R}_+; x^{2\gamma-1})$, respectively. In the latter case, the Parseval equality holds:

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1} dx = \frac{1}{2\pi} \int_{-\infty}^\infty |f^{\mathcal{M}}(\gamma + it)|^2 dt. \quad (1.8)$$

The Kontorovich–Lebedev transformation (1.4) (see [6]), in turn, is an isomorphism between the spaces $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ and $L_2(\mathbb{R}_+; x^{-1} dx)$ with the Parseval equality of the form

$$\int_0^\infty |[KLf](x)|^2 \frac{dx}{x} = \frac{\pi}{2} \int_0^\infty |f(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau. \quad (1.9)$$

The corresponding inverse operator in the mean-convergence sense in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ is written in the form

$$f(\tau) = \operatorname{l.i.m.}_{N \rightarrow \infty} \frac{2}{\pi |\Gamma(2i\tau)|^2} \int_{1/N}^N K_{2i\tau}(2\sqrt{x}) [KLf](x) \frac{dx}{x}. \quad (1.10)$$

2. THE PLANCHEREL THEOREM

Let us consider the Gamma-product transformation (1.1) in the weighted L_2 -spaces. Our goal is to prove an analogue of the Plancherel theorem for this transformation. Indeed, we have the following:

THEOREM 1. *Let $f(\tau) \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. The Gamma-product transform (1.1), where the integral converges in mean with respect to the norm in $L_2(\mathbb{R}; dx)$ forms the isomorphism*

$$\mathcal{G} : L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau) \leftrightarrow L_2(\mathbb{R}; dx),$$

where the reciprocal inverse operator is given by

$$f(\tau) = \text{l.i.m.}_{N \rightarrow \infty} P.V. \frac{1}{4\pi^2 |\Gamma(2i\tau)|^2} \int_{-N}^N \Gamma(-i(x+\tau)) \Gamma(i(\tau-x)) [\mathcal{G}f](x) dx \quad (2.1)$$

with the convergence with respect to the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. If $f, g \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$, then the Plancherel identity holds, i.e.,

$$\int_{-\infty}^{\infty} [\mathcal{G}f](x) \overline{[\mathcal{G}g](x)} dx = 4\pi^2 \int_0^{\infty} f(\tau) \overline{g(\tau)} |\Gamma(2i\tau)|^2 d\tau. \quad (2.2)$$

In particular, the Parseval equality holds, i.e.,

$$\int_{-\infty}^{\infty} |[\mathcal{G}f](x)|^2 dx = 4\pi^2 \int_0^{\infty} |f(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau. \quad (2.3)$$

Proof. We first suppose that $f \in C_0^\infty(\mathbb{R}_+)$, i.e., belongs to the space of smooth functions with compact support on $\mathbb{R}_+$. Hence, taking $\alpha > 0$, we introduce the function

$$\Phi_f(z) = \int_0^{\infty} \Gamma(z+i\tau) \Gamma(z-i\tau) f(\tau) d\tau, \quad (2.4)$$

where $z = \alpha + ix$. By the elementary inequality for Euler's Gamma-function [1] $|\Gamma(z)| \leq |\Gamma(\text{Re}z)|$ we get that integral (2.4) converges uniformly with respect to $z \in \mathbb{C}$, $0 < \alpha_0 \leq \alpha \leq A$, since

$$\begin{aligned} & \int_0^{\infty} |\Gamma(z+i\tau) \Gamma(z-i\tau) f(\tau)| d\tau \\ & \leq \Gamma^2(\alpha) \int_{\text{supp} f} |f(\tau)| d\tau < C_A \int_{\text{supp} f} |f(\tau)| d\tau < \infty, \end{aligned}$$

where $C_A > 0$ is a constant. Moreover, the integrand is analytic with respect to z and we can choose α_0 and A as arbitrarily small and large numbers, respectively. Consequently, this yields that $\Phi_f(z)$ is analytic in the right half-plane. Further, we invoke

integral (1.5) for the modified Bessel function $K_{2i\tau}(2\sqrt{x})$ and the inverse Mellin transform (1.7) to represent the product of Gamma-functions as (see (1.6))

$$\Gamma(z+i\tau)\Gamma(z-i\tau) = 2 \int_0^\infty K_{2i\tau}(2\sqrt{x})x^{z-1} dx. \quad (2.5)$$

Substituting (2.5) into (2.4), we change the order of integration via Fubini's theorem and write $\Phi_f(z)$ as a composition of the Kontorovich–Lebedev transform (1.4) and Mellin transform (1.6)

$$\Phi_f(z) = 2 \int_0^\infty [KLf](x)x^{z-1} dx. \quad (2.6)$$

Hence, by the Parseval equality (1.8) for the Mellin transform we obtain

$$\int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx = 8\pi \int_0^\infty |[KLf](t)|^2 t^{2\alpha-1} dt. \quad (2.7)$$

However, letting $\alpha \in (0, A)$ we appeal to the Parseval equality (1.9) and to the inequality $|K_{2i\tau}(2\sqrt{t})| \leq K_0(2\sqrt{t})$ (see [1], [6]) for the modified Bessel functions in order to majorize the integral on the right-hand side of (2.7) as

$$\begin{aligned} \int_0^\infty |[KLf](t)|^2 t^{2\alpha-1} dt &= \left(\int_0^1 + \int_1^\infty \right) |[KLf](t)|^2 t^{2\alpha-1} dt \\ &\leq \int_0^1 |[KLf](t)|^2 \frac{dt}{t} + \int_1^\infty K_0^2(2\sqrt{t}) t^{2A-1} dt \left(\int_{\text{supp } f} |f(\tau)| d\tau \right)^2 < \infty. \end{aligned} \quad (2.8)$$

Consequently, the left-hand side of (2.7) is bounded with respect to $\alpha \in (0, A)$ and $\Phi_f(z) \in \mathbb{H}_2^{(0,A)}$, where $\mathbb{H}_2^{(0,A)}$ denotes the Hardy space [5] of analytic functions in the strip $\text{Re } z \in (0, A)$ such that

$$\sup_{\alpha \in (0,A)} \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx < \infty.$$

Moreover, again invoking (2.7), we have

$$\int_{-\infty}^\infty |\Phi_f(\alpha_1 + ix) - \Phi_f(\alpha_2 + ix)|^2 dx = 8\pi \int_0^\infty |[KLf](t)|^2 (t^{\alpha_1} - t^{\alpha_2})^2 \frac{dt}{t}. \quad (2.9)$$

The right-hand side of Eq. (2.9) tends to zero as $\alpha_i \rightarrow 0+$, $i = 1, 2$, since via (2.8) and the dominated convergence theorem we can pass to the limit under the integral sign. Thus, $\Phi_f(\alpha + ix)$ converges in mean to some function $\Phi(ix)$ as $\alpha \rightarrow 0+$. Hence,

denoting its inverse Mellin transform by $\varphi(t) \in L_2(\mathbb{R}_+; t^{-1} dt)$, from (1.8) we find that

$$\begin{aligned} & \int_{-\infty}^{\infty} |\Phi_f(ix) - \Phi_f(\alpha + ix)|^2 dx \\ &= 2\pi \int_0^{\infty} |\varphi(t) - 2[KLf](t)t^\alpha|^2 \frac{dt}{t} \rightarrow 0, \quad \alpha \rightarrow 0+. \end{aligned} \quad (2.10)$$

By Fatou's lemma this implies the equality

$$\int_0^{\infty} |\varphi(t) - 2[KLf](t)|^2 \frac{dt}{t} = 0,$$

which means that $\varphi(t) = 2[KLf](t)$ for almost all $t \in \mathbb{R}_+$. Taking into account Eq. (1.9) we arrive at the following Parseval identities

$$\int_{-\infty}^{\infty} |\Phi_f(ix)|^2 dx = 8\pi \int_0^{\infty} |[KLf](t)|^2 \frac{dt}{t} = 4\pi^2 \int_0^{\infty} |f(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau. \quad (2.11)$$

If we prove that $\Phi_f(ix) = [\mathcal{G}f](x)$ for almost all $x \in \mathbb{R}$, then we establish (2.3) for any $f \in C_0^\infty(\mathbb{R}_+)$. Indeed, let $|x| \notin \text{supp } f$. Then appealing to the dominated convergence theorem, we immediately pass to the limit under the integral sign in (2.4) as $\alpha \rightarrow 0+$ to obtain

$$\begin{aligned} \Phi_f(ix) &= \lim_{\alpha \rightarrow 0+} \int_{\text{supp } f} \Gamma(\alpha + i(x + \tau)) \Gamma(\alpha + i(x - \tau)) f(\tau) d\tau \\ &= \int_{\text{supp } f} \Gamma(i(x + \tau)) \Gamma(i(x - \tau)) f(\tau) d\tau = [\mathcal{G}f](x). \end{aligned} \quad (2.12)$$

Otherwise, let $x > 0$ be such that $x \in \text{supp } f$ (the case $x < 0$ can be considered in the same way). By the definition of the Cauchy principal value we write integral (1.1) in the form

$$[\mathcal{G}f](x) = \lim_{\delta \rightarrow 0+} \left(\int_0^{x-\delta} + \int_{x+\delta}^{\infty} \right) \Gamma(i(x + \tau)) \Gamma(i(x - \tau)) f(\tau) d\tau. \quad (2.13)$$

Our goal is to prove that Eq. (2.12) holds in this case. In fact, it is easily seen that one can write (2.13) as the iterated limit

$$\begin{aligned} [\mathcal{G}f](x) &= \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \left(\int_0^{x-\delta} + \int_{x+\delta}^{\infty} \right) \Gamma(\alpha + i(x + \tau)) \Gamma(\alpha + i(x - \tau)) f(\tau) d\tau \\ &= \lim_{\alpha \rightarrow 0+} \Phi(\alpha + ix) - \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \int_{x-\delta}^{x+\delta} \Gamma(\alpha + i(x + \tau)) \Gamma(\alpha + i(x - \tau)) f(\tau) d\tau, \end{aligned}$$

and Eq. (2.12) will be true if we show that, for all $x > 0$,

$$I(x) = \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \int_{x-\delta}^{x+\delta} \Gamma(\alpha + i(x + \tau)) \Gamma(\alpha + i(x - \tau)) f(\tau) d\tau = 0.$$

Indeed, using the mean-value theorem, we have $f(x + t) - f(x) = tf'(x + t\xi(t))$, where $0 \leq \xi \leq 1$. Therefore,

$$\begin{aligned} I(x) &= \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \int_{-\delta}^{\delta} \Gamma(\alpha + i(2x + t)) \Gamma(\alpha - it) f(x + t) dt \\ &= \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \int_{-\delta}^{\delta} \Gamma(\alpha + i(2x + t)) \Gamma(\alpha - it) t f'(x + t\xi(t)) dt \\ &\quad + f(x) \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \int_{-\delta}^{\delta} \Gamma(\alpha + i(2x + t)) \Gamma(\alpha - it) dt = I_1(x) + I_2(x). \end{aligned}$$

It is straightforward to verify that $I_1(x) = 0$ for all $x > 0$. In fact, choosing $0 < \delta < 2x$ and invoking the relation $\Gamma(1 + z) = z\Gamma(z)$, we find, for $\alpha \in [0, 1]$,

$$\begin{aligned} &\int_{-\delta}^{\delta} |\Gamma(\alpha + i(2x + t)) \Gamma(\alpha - it) t f'(x + t\xi(t))| dt \\ &= \int_{-\delta}^{\delta} \left| \Gamma(1 + \alpha + i(2x + t)) \Gamma(1 + \alpha - it) \frac{t f'(x + t\xi(t))}{\sqrt{(\alpha^2 + (2x + t)^2)(\alpha^2 + t^2)}} \right| dt \\ &\leq \Gamma^2(1 + \alpha) \int_{-\delta}^{\delta} |f'(x + t\xi(t))| \frac{dt}{|2x + t|} \\ &\leq \int_{-\delta}^{\delta} |f'(x + t\xi(t))| \frac{dt}{2x - |t|} \\ &< 2\delta \frac{\sup_{|t| \leq \delta} |f'(x + t\xi(t))|}{2x - \delta} \rightarrow 0, \quad \delta \rightarrow 0+. \end{aligned}$$

Thus, passing first to the limit with respect to $\alpha \rightarrow 0+$ under the integral sign and then with respect to $\delta \rightarrow 0+$, we get $I_1(x) = 0$ for all $x > 0$. Considering $I_2(x)$, we use a representation for the product of Gamma-functions as the following cosine Fourier integral (see [6, relation (1.104)]):

$$\Gamma(\alpha + i(x + \tau)) \Gamma(\alpha + i(x - \tau)) = \frac{\Gamma(2(\alpha + ix))}{2^{2(\alpha + ix) - 1}} \int_0^{\infty} \frac{\cos \tau y}{\cosh^{2(\alpha + ix)}(y/2)} dy. \quad (2.14)$$

Hence, we obtain

$$I_2(x) = f(x) \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \int_{x-\delta}^{x+\delta} \Gamma(\alpha + i(x - \tau)) \Gamma(\alpha + i(x + \tau)) d\tau$$

$$= f(x) \frac{\Gamma(2ix)}{2^{2(ix-1)}} \lim_{\delta \rightarrow 0+} \lim_{\alpha \rightarrow 0+} \int_0^\infty \frac{\sin \delta y \cos xy}{y \cosh^{2(\alpha+ix)}(y/2)} dy,$$

where the change of the order of integration is justified by of the absolute and uniform convergence of the integral with respect to y for each $\alpha > 0$. However, the latter integral is uniformly convergent with respect to $\alpha \geq 0$ via the Abel test. Thus, passing to the limit under the integral sign as $\alpha \rightarrow 0+$, we have

$$I_2(x) = f(x) \frac{\Gamma(2ix)}{2^{2(ix-1)}} \lim_{\delta \rightarrow 0+} \int_0^\infty \frac{\sin \delta y \cos xy}{y \cosh^{2ix}(y/2)} dy.$$

It remains to show that the latter limit is zero. Indeed, splitting it into two integrals over $(0, 1)$ and $(1, \infty)$, we find that, in the first integral, we can easily pass to the limit as $\delta \rightarrow 0+$ via the absolute and uniform convergence, and this gives zero. Appealing again to the Abel test, we conclude that the second integral

$$\int_1^\infty \frac{\cos xy}{y} \frac{\sin \delta y}{\cosh^{2ix}(y/2)} dy$$

converges uniformly in $\delta \geq 0$ for each $x > 0$. Thus, it tends to zero as $\delta \rightarrow 0+$, and $I_2(x) = 0$ for all $x > 0$. Combining with the above arguments and returning to (2.12), we establish that $\Phi_f(ix) = [\mathcal{G}f](x)$ for almost all $x \in \mathbb{R}$. Consequently, the Parseval equality (2.3) holds for any $f \in C_0^\infty(\mathbb{R}_+)$. Since this set of functions is dense in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$, we extend Eq. (2.3) to the whole space. Hence, with the parallelogram identity we immediately establish the Plancherel equality (2.2) and we continuously extend the Gamma-product transform (1.1) onto the whole space $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. As in [7], we show that this transformation forms the isomorphism

$$\mathcal{G}: L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau) \leftrightarrow L_2(\mathbb{R}; dx)$$

and that its extension can be represented by the following limit with respect to the norm in $L_2(\mathbb{R}; dx)$:

$$[\mathcal{G}f](x) = \text{l.i.m.}_{N \rightarrow \infty} \text{P.V.} \int_0^N \Gamma(i(x+\tau)) \Gamma(i(x-\tau)) f(\tau) d\tau.$$

We show that reciprocal formula (2.1) follows from the Plancherel equality (2.2). Let us fix $\xi > 0$ and take

$$g(\tau) = \begin{cases} \tau, & \text{if } \tau \in [0, \xi], \\ 0, & \text{if } \tau \in (\xi, \infty), \end{cases}$$

which evidently belongs to the space $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. Hence, from (2.2) we arrive at the equality

$$4\pi^2 \int_0^\xi \tau f(\tau) |\Gamma(2i\tau)|^2 d\tau = \int_{-\infty}^\infty [\mathcal{G}f](x) \int_0^\xi \tau \Gamma(-i(x+\tau)) \Gamma(i(\tau-x)) d\tau dx. \quad (2.15)$$

Choosing $N \in \mathbb{N}$ sufficiently large and denoting

$$[\mathcal{G}f]_N(x) = \begin{cases} [\mathcal{G}f](x) & \text{if } x \in [-N, N], \\ 0 & \text{if } x \notin [-N, N], \end{cases}$$

we note that $\|[\mathcal{G}f] - [\mathcal{G}f]_N\|_{L_2(\mathbb{R})} \rightarrow 0$, $N \rightarrow \infty$. Moreover, by Parseval equality (2.3) the corresponding sequence $\{f_N\}$ converges to f , and from (2.15) we obtain

$$4\pi^2 \int_0^\xi \tau f_N(\tau) |\Gamma(2i\tau)|^2 d\tau = \int_{-N}^N [\mathcal{G}f](x) \int_0^\xi \tau \Gamma(-i(x+\tau)) \Gamma(i(\tau-x)) d\tau dx.$$

Since by the Cauchy–Schwarz inequality $\tau f_N(\tau) |\Gamma(2i\tau)|^2 \in L_1(0, \xi)$, this implies

$$f_N(\xi) = \frac{1}{4\pi^2 \xi |\Gamma(2i\xi)|^2} \frac{d}{d\xi} \int_{-N}^N [\mathcal{G}f](x) \int_0^\xi \tau \Gamma(-i(x+\tau)) \Gamma(i(\tau-x)) d\tau dx, \quad (2.16)$$

and we get that $f(\tau) = \text{l.i.m.}_{N \rightarrow \infty} f_N(\tau)$ with respect to the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$.

Meanwhile, fixing $N \in \mathbb{N}$, the order of integration in (2.16) can be inverted due to the familiar Poincaré–Bertrand formula. Hence, we derive

$$f_N(\xi) = \frac{1}{4\pi^2 \xi |\Gamma(2i\xi)|^2} \frac{d}{d\xi} \int_0^\xi \tau \int_{-N}^N [\mathcal{G}f](x) \Gamma(-i(x+\tau)) \Gamma(i(\tau-x)) dx d\tau. \quad (2.17)$$

However,

$$\begin{aligned} & \tau \int_{-N}^N [\mathcal{G}f](x) \Gamma(-i(x+\tau)) \Gamma(i(\tau-x)) dx \\ &= \tau \int_{-N}^N [\mathcal{G}f](x) \frac{\Gamma(1-i(x+\tau)) \Gamma(1+i(\tau-x))}{\tau^2 - x^2} dx \\ &= \frac{1}{2} \int_{-N}^N [\mathcal{G}f](x) \frac{\Gamma(1-i(x+\tau)) \Gamma(1+i(\tau-x))}{\tau+x} dx \\ & \quad - \frac{1}{2} \int_{-N}^N [\mathcal{G}f](x) \frac{\Gamma(1-i(x+\tau)) \Gamma(1+i(\tau-x))}{x-\tau} dx \\ &= \frac{1}{2} \int_{-N}^N [\mathcal{G}f](x) \frac{\Gamma(1-i(x+\tau)) \Gamma(1+i(\tau-x)) - \Gamma(1+2i\tau)}{\tau+x} dx \\ & \quad + \frac{\Gamma(1+2i\tau)}{2} \int_{-N}^N \frac{[\mathcal{G}f](x)}{\tau+x} dx \\ & \quad - \frac{1}{2} \int_{-N}^N [\mathcal{G}f](x) \frac{\Gamma(1-i(x+\tau)) \Gamma(1+i(\tau-x)) - \Gamma(1-2i\tau)}{x-\tau} dx \end{aligned}$$

$$-\frac{\Gamma(1-2i\tau)}{2} \int_{-N}^N \frac{[\mathcal{G}f](x)}{x-\tau} dx = J_1(\tau) + J_2(\tau) - J_3(\tau) - J_4(\tau).$$

Appealing to the Riesz theorem on the L_p -boundedness of the Hilbert transform (see Theorem 101 in [5]), we conclude that the functions $J_2(\tau)$, $J_4(\tau) \in L_1((0, \xi); d\tau)$. At the same time, by the mean-value theorem we easily get that $J_1(\tau)$, $J_3(\tau)$ are bounded and, therefore, belong to $L_1((0, \xi); d\tau)$ for each $\xi > 0$. Thus, differentiating in (2.17) with respect to ξ and passing to the limit by the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ as $N \rightarrow \infty$, we establish (2.1). This ends the proof of Theorem 1.

Remark 1. Via the asymptotics as $\tau \rightarrow +\infty$ of the ratio of Gamma-functions (see the Stirling formula, for instance, in [1], [6]) we have that, for each $x \in \mathbb{R}$,

$$\frac{\Gamma(i(x+\tau))\Gamma(i(x-\tau))}{\Gamma(2i\tau)} = O\left(\frac{1}{\sqrt{\tau}}\right), \quad \tau \rightarrow +\infty.$$

Hence, one can show that, for $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$, integral (1.1) generally does not exist. Nevertheless, according to Theorem 1, it converges in mean with respect to the norm in $L_2(\mathbb{R}; dx)$.

3. LIMIT VALUES IN HARDY'S SPACES

In this section, we describe a class of functions $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ having the corresponding Gamma-product transform $[\mathcal{G}f](x) \in L_2(\mathbb{R}; dx)$ as a limit almost everywhere as $\alpha \rightarrow 0+$ of the function (2.4) $\Phi_f(z) \in \mathbb{H}_2$, $z = \alpha + ix$. This means that Φ_f is analytic in the right half-plane $\alpha > 0$ and satisfies the condition

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx < \infty. \quad (3.1)$$

THEOREM 2. *Let $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. The Gamma-product transform $[\mathcal{G}f](x)$ is the limit of $\Phi_f(\alpha + ix) \in \mathbb{H}_2$ almost everywhere as $\alpha \rightarrow 0+$ if and only if the Kontorovich–Lebedev transform (1.4) $[KLf](x)$ is equal to zero for almost all $x > 1$.*

Proof. Necessity. The necessity immediately follows from Eq. (2.7). Indeed, since its left-hand side remains bounded as $\alpha \rightarrow \infty$, we have, for all $A > 1$,

$$\begin{aligned} \int_A^\infty |[KLf](t)|^2 \frac{dt}{t} &\leq A^{-2\alpha} \int_A^\infty |[KLf](t)|^2 t^{2\alpha-1} dt \\ &\leq \frac{A^{-2\alpha}}{8\pi} \sup_{\alpha>0} \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx = \text{const.} \quad A^{-2\alpha} \rightarrow 0, \quad \alpha \rightarrow \infty. \end{aligned}$$

Therefore,

$$\int_A^\infty |[KLf](t)|^2 \frac{dt}{t} = 0$$

and $[KLf](x) = 0$ for almost all $x > 1$. Since, as $\alpha \rightarrow 0+$, the left-hand side of (2.7) is bounded we find via Fatou's lemma that $[KLf](x) \in L_2((0, 1); x^{-1} dx)$.

Sufficiency. Conversely, let $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ be such that $[KLf](x) = 0$ for almost all $x > 1$. Hence, by (2.6) we put

$$\Phi_f(\alpha + ix) = 2 \int_0^1 [KLf](t) t^{\alpha+ix-1} dt, \quad \alpha > 0.$$

Since by the Cauchy–Schwarz inequality

$$\begin{aligned} \int_0^1 |[KLf](t)| t^{\alpha-1} dt &\leq \left(\int_0^1 |[KLf](t)|^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^1 t^{2\alpha-1} dt \right)^{1/2} \\ &= \frac{1}{\sqrt{2\alpha}} \left(\int_0^1 |[KLf](t)|^2 \frac{dt}{t} \right)^{1/2} < \infty, \end{aligned}$$

we have that $\Phi_f(\alpha + ix)$ is analytic in the right half-plane and, invoking Eq. (2.7), we obtain

$$\int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx = 8\pi \int_0^1 |[KLf](t)|^2 t^{2\alpha-1} dt \leq 8\pi \int_0^1 |[KLf](t)|^2 \frac{dt}{t} < \infty.$$

Thus, it satisfies condition (3.1) and, according to the theory of Hardy's spaces, $\Phi_f(\alpha + ix)$ attains, in the mean sense and for almost all $x \in \mathbb{R}$, its limit value in $\mathbb{H}_2 \ominus_f(x)$ as $\alpha \rightarrow 0+$.

However, if we take a sequence $\{f_n\}_{n=1}^{\infty}$ of $C_0^{\infty}(\mathbb{R}_+)$ -functions, which converges to f with respect to the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$, then (see (2.6))

$$\Phi_{f_n}(\alpha + ix) = 2 \int_0^{\infty} [KLf_n](t) t^{\alpha+ix-1} dt,$$

and, similarly, we derive that, for almost all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $\Phi_{f_n}(\alpha + ix) \rightarrow \Theta_{f_n}(x)$ as $\alpha \rightarrow 0+$. Consequently, by Theorem 1 we get that $\Theta_{f_n}(x) = [\mathcal{G}f_n](x)$ for almost all $x \in \mathbb{R}$. Furthermore, by (2.3) we have

$$\begin{aligned} \int_{-\infty}^{\infty} |[\mathcal{G}f](x) - [\mathcal{G}f_n](x)|^2 dx &= \int_{-\infty}^{\infty} |[\mathcal{G}(f - f_n)](x)|^2 dx \\ &= 4\pi^2 \int_0^{\infty} |f(\tau) - f_n(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.2)$$

On the other hand, invoking (1.9), we find

$$\int_{-\infty}^{\infty} |\Theta_f(x) - \Theta_{f_n}(x)|^2 dx = \lim_{\alpha \rightarrow 0+} \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix) - \Phi_{f_n}(\alpha + ix)|^2 dx$$

$$\begin{aligned}
 &= \lim_{\alpha \rightarrow 0+} 8\pi \int_0^\infty |[KLf](t) - [KLf_n](t)|^2 t^{2\alpha-1} dt \\
 &= \lim_{\alpha \rightarrow 0+} 8\pi \left[\int_0^1 |[KL(f - f_n)](t)|^2 t^{2\alpha-1} dt + \int_1^\infty |[KLf_n](t)|^2 t^{2\alpha-1} dt \right] \\
 &= 8\pi \left[\int_0^1 |[KL(f - f_n)](t)|^2 \frac{dt}{t} + \int_1^\infty |[KLf_n](t)|^2 \frac{dt}{t} \right] \\
 &\rightarrow 8\pi \int_1^\infty |[KLf](t)|^2 \frac{dt}{t}, \quad n \rightarrow \infty.
 \end{aligned}$$

Moreover under the conditions of the theorem, the latter integral is zero. Thus, combining this with (3.2), we obtain $\Theta_f(x) = [\mathcal{G}f](x)$ for almost all $x \in \mathbb{R}$. This completes the proof of Theorem 2.

We establish now a generalization of Theorem 2.

THEOREM 3. *Let $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. Let also $\Phi_f(\alpha + ix)$ be analytic in the right half-plane $\alpha > 0$ and satisfy the condition*

$$\int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx = O(y^{2\alpha}), \quad y > 0. \quad (3.3)$$

The Gamma-product transform $[\mathcal{G}f](x)$ is the limit of $\Phi_f(\alpha + ix)$ almost everywhere as $\alpha \rightarrow 0+$ if and only if the Kontorovich–Lebedev transform $[KLf](x)$ is equal to zero for almost all $x > y$. Besides, if y is the least of such numbers, then

$$\lim_{\alpha \rightarrow \infty} \frac{1}{2\alpha} \log \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx = \log y. \quad (3.4)$$

Proof. The first part of the theorem can be proved by previous Theorem 2. Indeed, denoting by $\Psi_f(z) = y^{-z} \Phi_f(z)$, we invoke condition (3.3) to derive

$$\int_{-\infty}^\infty |\Psi_f(\alpha + ix)|^2 dx = y^{-2\alpha} \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx = O(1).$$

Hence, $\Psi_f(z) \rightarrow \Psi_f(ix)$, $\alpha \rightarrow 0+$, for almost all $x \in \mathbb{R}$. Moreover, since (see (2.7))

$$\begin{aligned}
 y^{-2\alpha} \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx &= 8\pi y^{-2\alpha} \int_0^\infty |[KLf](t)|^2 t^{2\alpha-1} dt \\
 &= 8\pi \int_0^\infty |[KLf](ty)|^2 t^{2\alpha-1} dt,
 \end{aligned}$$

it again follows that, for $A > 1$,

$$\begin{aligned} \int_A^\infty |[K L f](ty)|^2 \frac{dt}{t} &= \int_{Ay}^\infty |[K L f](t)|^2 \frac{dt}{t} \\ &\leq A^{-2\alpha} \int_A^\infty |[K L f](ty)|^2 t^{2\alpha-1} dt \leq \text{const.} A^{-2\alpha} \rightarrow 0, \quad \alpha \rightarrow \infty. \end{aligned}$$

Therefore, $[K L f](x)$ is zero for almost all $x > y$. In the same way as in Theorem 2, we prove the sufficiency of this condition.

Further, we have

$$\begin{aligned} \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx &= 8\pi \int_0^y |[K L f](t)|^2 t^{2\alpha-1} dt \\ &\leq 8\pi y^{2\alpha} \int_0^\infty |[K L f](t)|^2 \frac{dt}{t}. \end{aligned} \quad (3.5)$$

Thus,

$$\frac{1}{2\alpha} \log \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx \leq \log y + \frac{1}{2\alpha} \log \left(8\pi \int_0^\infty |[K L f](t)|^2 \frac{dt}{t} \right).$$

Passing to the limit as $\alpha \rightarrow \infty$ in the latter inequality, we get

$$\lim_{\alpha \rightarrow \infty} \frac{1}{2\alpha} \log \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx \leq \log y. \quad (3.6)$$

On the other hand, letting

$$\rho(t) = \int_t^y |[K L f](u)|^2 \frac{du}{u}$$

and taking sufficiently small positive δ , after integration by parts we arrive at the estimate

$$8\pi \int_0^y |[K L f](t)|^2 t^{2\alpha-1} dt = 16\pi\alpha \int_0^y \rho(t) t^{2\alpha-1} dt \quad (3.7)$$

$$\geq 16\pi\alpha\rho(y-\delta) \int_0^{y-\delta} t^{2\alpha-1} dt = 8\pi\rho(y-\delta) (y-\delta)^{2\alpha}. \quad (3.8)$$

Thus, combining with (3.5), we take logarithm of the both sides of the latter inequality and then divide by 2α . Hence,

$$\frac{1}{2\alpha} \log \int_{-\infty}^\infty |\Phi_f(\alpha + ix)|^2 dx \geq \log(y-\delta) + \frac{1}{2\alpha} \log(8\pi\rho(y-\delta)).$$

Passing to the limit as $\alpha \rightarrow \infty$, we obtain

$$\lim_{\alpha \rightarrow \infty} \frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx \geq \log(y - \delta).$$

Since $\delta > 0$ is an arbitrary small number, this gives

$$\lim_{\alpha \rightarrow \infty} \frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx \geq \log y.$$

Taking into account inequality (3.6), we get (3.4). This proves the second part of the theorem. Theorem 3 is proved.

Acknowledgments

This work was supported in part by the “Centro de Matemática” of the University of Porto.

REFERENCES

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, vol. I, McGraw-Hill, New York, London, Toronto (1953).
2. A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series: Special Functions*, Gordon and Breach, New York and London (1986).
3. A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series: More Special Functions*, Gordon and Breach, New York and London (1989).
4. I. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York (1972).
5. E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford (1937).
6. S.B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong (1996).
7. S.B. Yakubovich, On the integral transformation associated with the product of Gamma-functions, *Portugaliae Mathematica*, **60**(3), 337–351 (2003).

REZIUOMĖ

S.B. Yakubovich. Gamma-sandaugos transformacijos L_2 -aprėžtumas

Nagrinėjama singuliari integralinė transformacija

$$[\mathcal{G}f](x) = \text{P.V.} \int_0^{\infty} \Gamma(i(x + \tau))\Gamma(i(x - \tau))f(\tau) d\tau, \quad x \in \mathbb{R}.$$

Nustatytos šios transformacijos analizinio pratęsimo į dešiniąją pusplėkštumą būtinos ir pakankamos sąlygos.